

# EMBEDDINGS OF GENERAL CURVES IN PROJECTIVE SPACES: THE RANGE OF THE QUADRICS

E. BALLICO

ABSTRACT. Let  $C \subset \mathbb{P}^r$  a general embedding of prescribed degree of a general smooth curve with prescribed genus. Here we prove that either  $h^0(\mathbb{P}^r, \mathcal{I}_C(2)) = 0$  or  $h^1(\mathbb{P}^r, \mathcal{I}_C(2)) = 0$  (a problem called the Maximal Rank Conjecture in the range of quadrics).

## 1. INTRODUCTION

Let  $C \subset \mathbb{P}^r$  be any projective curve. The curve  $C$  is said to have *maximal rank* if for every integer  $x > 0$  the restriction map  $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \rightarrow H^0(C, \mathcal{O}_C(x))$  has maximal rank, i.e. either it is injective or it is surjective.

For any curve  $X$  and any spanned  $L \in \text{Pic}(X)$  let  $h_L : X \rightarrow \mathbb{P}^r$ ,  $r := h^0(X, L) - 1$ , denote the morphism induced by the complete linear system  $|L|$ . Here we prove the following result, which improves one of the results in [12].

For all integers  $g, r, d$  set  $\rho(g, r, d) := (r + 1)d - rg - r(r + 1)$  (the Brill-Noether number for  $g_d^r$ 's on a curve of genus  $g$ ). Fix integers  $r \geq 3$ , and  $g \geq 3$ . Fix a general  $X \in \mathcal{M}_g$ . Brill-Noether theory says that  $G_d^r(X) \neq \emptyset$  if and only if  $\rho(g, r, d) \geq 0$  (equivalently,  $W_d^r(X) \neq \emptyset$  if and only if  $\rho(g, r, d) \geq 0$ ) ([1], Ch. V). The Maximal Rank Conjecture in  $\mathbb{P}^r$  asks if a general embedding in  $\mathbb{P}^r$  of a general curve has maximal rank. Since this is true for non-special embeddings ([3] if  $r = 3$ , [5] if  $r > 3$ ), we only need to consider triples  $(g, r, d)$  with  $d < g + r$  and  $\rho(g, r, d) \geq 0$ . For these triples of integers Brill-Noether theory gives  $W_d^r(X) \neq \emptyset$ , that  $W_d^r(X)$  has pure dimension  $\rho(g, r, d)$  (it is also irreducible if  $\rho(g, r, d) > 0$ ) and that  $W_d^r(X) \neq W_d^{r+1}(X)$ , i.e.  $h^0(X, L) = r + 1$  for a general  $L \in W_d^r(X)$  ([1], Ch. V) (in the case  $\rho(g, r, d) = 0$  we have  $W_d^{r+1}(X) = \emptyset$ ). Hence  $h^1(X, L) = r + 1$  for a general  $L \in W_d^r(X)$  (or for all  $L \in W_d^r(X)$  if  $\rho(g, r, d) = 0$ ). For this range of triples  $(g, r, d)$  it is very easy to prove that a general  $L \in W_d^r(X)$  is very ample (e.g., see the proof of [10], Theorem at pages 26-27).

In this paper we answer a question raised in [12] (J. Wang called it the Maximal Rank Conjecture for quadrics).

**Theorem 1.** *Fix integers  $g \geq 3$ ,  $r \geq 2$  and  $d$  such that  $\rho(g, r, d) \geq 0$  and  $d \leq g + r$ . Fix a general  $L \in W_d^r(X)$ . Then the symmetric multiplication rank  $\mu_L : S^2(H^0(X, L)) \rightarrow H^0(X, L^{\otimes 2})$  has maximal rank, i.e. it is either injective or surjective.*

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Since  $\mu_L$  is obviously injective if  $r = 2$ , to prove Theorem 1 we may assume  $r \geq 3$ . The surjectivity part in Theorem 1 is true ([8], Theorem 1); this part corresponds to the triples  $(g, r, d)$  with  $r \geq 3$ ,  $0 \leq g + r - d \leq (r - 1)/2$  and  $(r + 1)(g + r - d) \leq g \leq r(r - 1)/2 + 2(g + r - d)$ .

Fix a general  $X \in \mathcal{M}_g$ . M. Teixidor i Bigas proved that if  $d \leq g + 1$ , then  $\mu_L$  is injective for all  $L \in \text{Pic}^d(X)$ . See [4] and [2] for the Maximal Rank Conjecture for  $r \leq 4$ .

We work over an algebraically closed field with characteristic zero.

## 2. PROOF OF THEOREM 1

**Lemma 1.** *Fix a set  $S \subset \mathbb{P}^r$ ,  $r \geq 3$ , such that  $\sharp(S) = r + 2$  and  $S$  is in linearly general position, i.e. any  $r + 1$  of its points span  $\mathbb{P}^r$ . For each  $P \in S$  let  $L_P \subset \mathbb{P}^r$  be a line such that  $P \in L_P$ . Fix any closed subset  $T \subset \mathbb{P}^r$  such that  $\dim(T) \leq 1$ . Then there exists a rational normal curve  $D \subset \mathbb{P}^r$  such that  $S \subset D$ , for each  $P \in S$  the line  $L_P$  is not the tangent line of  $D$  at  $P$  and  $D \cap (T \setminus T \cap S) = \emptyset$ .*

*Proof.* Let  $E(S)$  be the set of all rational normal curves  $C \subset \mathbb{P}^r$  containing  $S$ . For each  $Q \in (\mathbb{P}^r \setminus S)$  set  $E(S, Q) := \{C \in E(S) : Q \in C\}$ . If  $S \cup \{Q\}$  is in linearly general position, then  $\sharp(E(S, Q)) = 1$ . If  $S \cup \{Q\}$  is not in general position, then  $E(S, Q) = \emptyset$ . Hence  $E(S)$  is a quasi-projective irreducible variety of dimension  $r - 1$ . For each  $P \in S$  and any tangent vector of  $\mathbb{P}^r$  at  $P$  set  $E(\nu) := \{C \in E(S) : \nu \text{ is tangent to } C \text{ at } P\}$ . Either  $E(\nu) = \emptyset$  or  $E(\nu)$  is a single point (part (b) of [9], Theorem 1). Since  $\dim(\cup_{Q \in (T \setminus T \cap S)} E(S, Q)) \leq \dim(T) = 1$ , a general  $D \in E(S)$  satisfies the thesis of the lemma.  $\square$

*Proof of Theorem 1.* Fix integers  $x, y$  such that  $y \geq 0$  and either  $x \geq y + r$  or  $y \geq 2$  and  $x \geq r$  and  $y \leq x - r + \lfloor (x - r - 2)/(r - 2) \rfloor$ . In the papers [4] (case  $r = 3$ ) and [6] (case  $r \geq 4$ ) the authors defined an irreducible component  $W(x, y; r)$  of the Hilbert scheme of  $\mathbb{P}^r$  whose general member is a non-degenerate smooth curve of genus  $y$  and degree  $x$ . If  $x \geq r + y$ , this is the component of  $\text{Hilb}(\mathbb{P}^r)$  whose general element is a non-special curve, while if  $x < y + r$ , then  $h^1(E, \mathcal{O}_E(1)) = y + r - x$  and  $E$  is linearly normal. If  $\rho(y, r, x) \geq 0$ , then a general  $E \in W(x, y; r)$  has general moduli ([6], Proposition 3.1). Since  $\mu_L$  is obviously injective if  $r = 2$ , to prove Theorem 1 we may assume  $r \geq 3$ . Hence we may assume that  $L$  is very ample (e.g., see the proof of [10], Theorem at pages 26-27). Since  $W_d^r(X)$  is irreducible, the semicontinuity theorem for cohomology gives that to prove Theorem 1 for the triple  $(g, r, d)$  it is sufficient to find one  $L \in W_d^r(X) \setminus W_d^{r+1}(X)$  such that  $\mu_L$  has maximal rank. Since  $h^0(X, L) = r + 1$ , we have  $\dim S^2(H^0(X, L)) = \binom{r+2}{2}$ . By Gieseker-Petri theory we have  $h^1(X, L^{\otimes 2}) = 0$ . Hence  $h^0(X, L^{\otimes 2}) = 2d + 1 - g$ . The surjectivity part in Theorem 1 is true ([8], Theorem 1). Hence we may assume  $2d + 1 - g > \binom{r+2}{2}$ . Set  $c := g + r - d$ . Riemann-Roch gives  $c = h^1(X, L)$ . Since a general non-special embedding of  $X$  has maximal rank ([3] for  $r = 3$ , [4] for  $r \geq 4$ ), we may assume  $c > 0$ . Since  $\rho(d, g, r) = (r + 1)(g + r - c) - rg - r(r + 1)$ , the assumption  $\rho(g, r, d) \geq 0$  is equivalent to  $g \geq c(r + 1)$ . For all integers  $t \geq 3$  and  $b \geq 0$  set  $g_{t,b} := t(t - 1)/2 + 2b$ . Notice that

$$(1) \quad 2(g_{r,b} + r - b) + 1 - g_{r,b} = \binom{r+2}{2}$$

We have  $\rho(g_{r,b}, r, g_{r,b} + r - b) = g_{r,b} - b(r + 1)$ . Hence  $\rho(g_{r,b}, r, g_{r,b} + r - b) \geq 0$  if and only if  $g_{r,b} \geq b(r + 1)$ . Hence  $\rho(g_{r,b}, r, g_{r,b} + r - b) \geq 0$  if and only if  $b \leq r/2$ .

(a) Here we assume  $c \leq r/2$ . Hence  $\rho(g_{r,c}, r, g_{r,c} + r - c) \geq 0$ . Fix a general  $E \in \mathcal{M}_g$  and a general  $R \in W_{g_{r,c}+r-c}^r(E)$ . Brill-Noether theory gives  $h^0(E, R) = r + 1$ . By [8], Theorem 1,  $R$  is very ample and the curve  $h_R(E)$  is projectively normal. Hence  $\mu_R$  is surjective. The case  $b = c$  of (1) gives that  $\mu_R$  is bijective. Hence  $h^0(\mathbb{P}^r, \mathcal{I}_{h_R(E)}(2)) = 0$ . Since  $E$  has general moduli, we have  $h_R(E) \in W(g_{r,c} + r - c, g_{r,c}; r)$ . Since  $2(g + r - c) + 1 - g > \binom{r+2}{2}$ , we have  $g > g_{r,c}$ . We have  $d = g + r - c$  and hence  $g - g_{r,c} = d - (g_{r,c} + r - c)$ . Let  $A \subset \mathbb{P}^r$  be the union of  $h_R(E)$  and  $g - g_{r,c}$  general secant lines of  $h_R(E)$ . We have  $A \in W(d, g; r)$  (apply  $c$  times [6], Lemma 2.2). Since  $A \supset E$  and  $h^0(\mathbb{P}^r, \mathcal{I}_{h_R(E)}(2)) = 0$ , we have  $h^0(\mathbb{P}^r, \mathcal{I}_A(2)) = 0$ . Hence  $h^0(\mathbb{P}^r, \mathcal{I}_C(2)) = 0$  for a general  $C \in W(d, g; r)$ . Since  $\rho(d, g, r) \geq 0$ ,  $C$  is a linearly normal curve of degree  $d$  and genus  $g$  with general moduli. By semicontinuity we get the injectivity of  $\mu_L$  for a general  $X$  and a general  $L$ .

(b). From now on (i.e. in steps (b), (c), (d), (e)) we assume  $c > r/2$ . In this step we assume  $r$  even. Notice that  $g_{r,r/2} = r(r + 1)/2$ . Hence  $\rho(g_{r,r/2}, g_{r,r/2} + r - r/2, r) = 0$ ,  $W(g_{r,r/2} + r/2, g_{r,r/2}; r)$  is defined and a general element of it has general moduli. Fix a general  $Y \in W(g_{r,r/2} + r/2, g_{r,r/2}; r)$ . Since  $Y$  has general moduli, [8], Theorem 1, and (1) give  $h^i(\mathbb{P}^r, \mathcal{I}_Y(2)) = 0$ ,  $i = 0, 1$ . Set  $k := g - c(r + 1)$ . Fix a general  $S \subset Y$  such that  $\sharp(S) = (r + 2)(c - r/2)$  and take a partition of  $S$  into  $c - r/2$  disjoint sets  $S_i$ ,  $1 \leq i \leq c - r/2$ , such that  $\sharp(S_i) = r + 2$  for all  $i$ . Let  $E \subset \mathbb{P}^r$  be a general union of  $Y$ ,  $c - r/2$  rational normal curves  $E_i$ ,  $1 \leq i \leq c - r/2$ , such that  $S_i \subset E_i$  for all  $i$  and  $k$  general secant lines  $R_j$ ,  $1 \leq j \leq k$ . We may find these rational normal curves and these lines so that  $D_i \cap R_j = \emptyset$  for all  $i, j$ ,  $S_i = D_i \cap Y$  for all  $i$ , each  $D_i$  intersects quasi-transversally  $Y$ ,  $D_i \cap D_h = \emptyset$  for all  $i \neq h$ ,  $R_j \cap R_k = \emptyset$  for all  $j \neq k$ , each  $R_j$  intersects  $Y$  quasi-transversally and  $\sharp(R_i \cap Y) = 2$  for all  $i$  (first add the  $k$  general secant lines and then use Lemma 1). Notice that  $E$  is a nodal curve of degree  $d$  and arithmetic genus  $g$ . Since  $E \supseteq Y$ , we have  $h^0(\mathbb{P}^r, \mathcal{I}_E(2)) = 0$ . By [6], Lemmas 2.2 and 2.3, we have  $E \in W(d, g; r)$ . By semicontinuity we have  $h^0(\mathbb{P}^r, \mathcal{I}_F(2)) = 0$  for a general  $F \in W(d, g; r)$ . Since  $\rho(g, r, d) \geq 0$ ,  $F$  has general moduli ([6], Proposition 3.1).

(c) From now on we assume  $r$  odd. Since the case  $r = 3$  is true (e.g. by [12], Theorem 1.6, or by [4], Theorem 1), we assume  $r \geq 5$ .

(d) In this step we prove the existence of  $Y_2 \in W((r^2 + 2r + 1)/2, (r^2 + r + 2)/2; r)$  such that  $h^i(\mathbb{P}^r, \mathcal{I}_{Y_2}(2)) = 0$ ,  $i = 0, 1$ . Notice that  $g_{r,(r+1)/2} = r(r - 1)/2 + r + 1 = (r^2 + r + 2)/2$  and  $g_{r,(r+1)/2} + r - (r + 1)/2 = (r^2 + 2r + 1)/2$ . Hence  $2((r^2 + 2r + 1)/2) + 1 - (r^2 + r + 2)/2 = \binom{r+2}{2}$ . The irreducible component  $W((r^2 + 2r + 1)/2, (r^2 + r + 2)/2; r)$  of  $\text{Hilb}(\mathbb{P}^r)$  is defined, because  $(r^2 + 2r + 1)/2 \geq r$ ,  $(r^2 + r + 2)/2 \geq 2$  and  $(r^2 + r + 2)/2 \leq (r^2 + 1)/2 + \lfloor (r^2 - 3)/(2r - 4) \rfloor$  (the latter inequality is equivalent to the inequality  $(r + 1)(r - 2) \leq r^2 - 3$ ). Notice that  $\rho((r^2 + r + 2)/2, r, (r^2 + 2r + 1)/2) < 0$  and hence this case corresponds to a case with  $d' - g' = c = (r + 1)/2$ , but in a range of triples  $(d', g', r)$  for which there is no curve with general moduli. Fix a hyperplane  $H \subset \mathbb{P}^r$ . By [8], Theorem 1, applied in  $H$  there is a smooth curve  $Y_1 \subset H$  such that  $Y_1 \in W(g_{r-1,(r-1)/2} + (r - 1)/2, g_{r-1,(r-1)/2}; r - 1)$  and  $h^i(H, \mathcal{I}_{Y_1}(2)) = 0$ ,  $i = 0, 1$ . We have  $g_{r-1,(r-1)/2} = (r - 1)r/2$ ,  $g_{r-1,(r-1)/2} + (r - 1)/2 = (r^2 - 1)/2$  and  $\rho(g_{r-1,(r-1)/2}, r - 1, g_{r-1,(r-1)/2} + (r - 1)/2) = 0$ . Fix a general  $S \subset Y_1$  such that  $\sharp(S) = r + 1$ . Let  $B \subset \mathbb{P}^r$  be a smooth and linearly normal elliptic curve such that

$B \cap H = S$  ( $B$  exists, because any two subsets of  $H$  with cardinality  $r+1$  and in linearly general position are projectively equivalent). Set  $Y_2 := Y_1 \cup B$ . The curve  $Y_2$  is a connected and nodal curve with degree  $(r^2 + 2r + 1)/2$  and arithmetic genus  $(r^2 + r + 2)/2$ . By [8], Lemma 7, we have  $Y_2 \in W((r^2 + 2r + 1)/2, (r^2 + r + 2)/2; r)$ . Since  $B$  is an elliptic curve of degree  $\sharp(S)$ , we have  $h^0(B, \mathcal{O}_B(2)(-S)) = 0$ . Hence the Mayer-Vietoris exact sequence

$$0 \rightarrow \mathcal{O}_{Y_2}(2) \rightarrow \mathcal{O}_{Y_1}(2) \oplus \mathcal{O}_B(2) \rightarrow \mathcal{O}_S(2) \rightarrow 0$$

gives  $h^1(Y_2, \mathcal{O}_{Y_2}(2)) = 0$ . Since  $2 \cdot \deg(Y_2) + 1 - p_a(Y_2) = \binom{r+2}{2}$ , we have  $h^1(\mathbb{P}^r, \mathcal{I}_{Y_2}(2)) = h^0(\mathbb{P}^r, \mathcal{I}_{Y_2}(2))$ . Fix  $f \in H^0(\mathbb{P}^r, \mathcal{I}_{Y_2}(2))$ . Since  $f|_H$  vanishes on  $Y_1$  and  $h^0(H, \mathcal{I}_{Y_1}(2)) = 0$ ,  $f$  is divided by the equation  $z$  of  $H$ . Hence  $f/z \in H^0(\mathbb{P}^r, \mathcal{I}_B(2))$ . Since  $B$  spans  $\mathbb{P}^r$ , we get  $f/z = 0$ , i.e.  $f = 0$ . Hence  $h^i(\mathbb{P}^r, \mathcal{I}_{Y_2}(2)) = 0$ ,  $i = 0, 1$ .

(e) Here we assume  $r \geq 5$ ,  $r$  odd,  $c = (r+1)/2$  and  $g = (r^2 + 2r + 1)/2$ . Hence  $d = (r+1)^2/2 + (r-1)/2 = (r^2 + 3r)/2$ . Let  $Y_3$  be a general union of  $Y_2$  and  $(r-1)/2$  secant lines of  $Y_3$ . Since  $h^0(\mathbb{P}^r, \mathcal{I}_{Y_2}(2)) = 0$ , we have  $h^0(\mathbb{P}^r, \mathcal{I}_{Y_3}(2)) = 0$ . Since  $Y_2 \in W((r^2 + 2r + 1)/2, (r^2 + r + 2)/2; r)$ , we have  $Y_3 \in W((r^2 + 3r)/2, (r^2 + 2r + 1)/2; r)$  ([6], Lemma 2.2). By semicontinuity we have  $h^0(\mathbb{P}^r, \mathcal{I}_{Y_4}(2)) = 0$  for a general  $Y_4 \in W((r^2 + 3r)/2, (r^2 + 2r + 1)/2; r)$ . Since  $\rho((r^2 + 2r + 1)/2, r, (r^2 + 3r)/2) = 0$ ,  $Y_4$  has general moduli.

(f) In this step we assume  $r$  odd,  $r \geq 5$  and  $(c, g) \neq ((r+1)/2, (r+1)^2/2)$ , i.e. in this step we prove all the cases not yet proven. As in step (b) set  $k := g - c(r+1)$ . Fix a general  $S \subset Y$  such that  $\sharp(S) = (r+2)(c - r/2)$  and take a partition of  $S$  into  $c - r/2$  disjoint sets  $S_i$ ,  $1 \leq i \leq c - (r+1)/2$ , such that  $\sharp(S_i) = r+2$  for all  $i$ . Let  $E_4 \subset \mathbb{P}^r$  be a general union of  $Y_4$ ,  $c - (r+1)/2$  rational normal curves  $D_i$ ,  $1 \leq i \leq c - (r+1)/2$ , such that  $S_i \subset D_i$  for all  $i$  and  $k$  general secant lines  $R_j$ ,  $1 \leq j \leq k$ . We may find these rational normal curves and these lines so that  $D_i \cap R_j = \emptyset$  for all  $i, j$ ,  $S_i = D_i \cap Y$  for all  $i$ , each  $D_i$  intersects quasi-transversally  $Y_4$ ,  $D_i \cap D_h = \emptyset$  for all  $i \neq h$ ,  $R_j \cap R_k = \emptyset$  for all  $j \neq k$ , each  $R_j$  intersects  $Y$  quasi-transversally and  $\sharp(R_i \cap Y) = 2$  for all  $i$ . Since  $h^0(\mathbb{P}^r, \mathcal{I}_{E_4}(2)) \leq h^0(\mathbb{P}^r, \mathcal{I}_{Y_4}(2)) = 0$ , we conclude as in step (b).  $\square$

## REFERENCES

- [1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of Algebraic curves*, I, Springer, Berlin, 1985.
- [2] E. Ballico, On the Maximal Rank Conjecture in  $\mathbb{P}^4$ , *Int. J. Pure Appl. Math.* 53 (2009), no. 3, 363–376.
- [3] E. Ballico and Ph. Ellia, The maximal rank conjecture for nonspecial curves in  $\mathbb{P}^3$ , *Invent. Math.* 79 (1985), no. 3, 541–555.
- [4] E. Ballico and Ph. Ellia, Beyond the maximal rank conjecture for curves in  $\mathbb{P}^3$ , in: *Space Curves*, Proceedings Rocca di Papa, pp. 1–23, Lecture Notes in Math. 1266, Springer, Berlin, 1985.
- [5] E. Ballico and Ph. Ellia, The maximal rank conjecture for non-special curves in  $\mathbb{P}^n$ , *Math. Z.* 196 (1987), 355–367.
- [6] E. Ballico and Ph. Ellia, On the existence of curves with maximal rank in  $\mathbb{P}^n$ , *J. Reine Angew. Math.* 397 (1989), 1–22.
- [7] E. Ballico and C. Fontanari, Normally generated line bundles on general curves, *J. Pure Appl. Algebra* 214 (2010), no. 6, 837–840.
- [8] E. Ballico and C. Fontanari, Normally generated line bundles on general curves, II, *J. Pure Appl. Algebra* 214 (2010), no. 8, 1450–1455.
- [9] D. Eisenbud and J. Harris, Finite projective schemes in linearly general position, *J. Algebraic Geom.* 1 (1992), no. 1, 15–30.

- [10] E. Sernesi, On the existence of certain families of curves, *Invent. Math.* 75 (1984), no. 1, 25–57.
- [11] M. Teixidor i Bigas, Injectivity of the symmetric map for line bundles, *Manuscripta Math.* 112 (2003), 511–517.
- [12] J. Wang, On the generic vanishing of certain cohomology groups, arXiv:1108.4714.

DEPT. OF MATHEMATICS, UNIVERSITY OF TRENTO, 38123 POVO (TN), ITALY  
*E-mail address:* `ballico@science.unitn.it`